Exploration of Mathematical Thinking and its Development through Geogebra

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Abstract

The purpose of this study was to know about the potential effect of geogebra software on students' mathematical thinking and to explore thinking structure in analytic geometry. To investigate geogebra effect, a sample of forty students (grade-12) of F.G Boys Inter College Mardan Cantt was selected. Further, they were divided on the basis of their grade-11 scores record into experimental and control group. Two groups with almost equal statistical background were constructed through pair random sampling and with the same compatibility in the biological age and the social background. A six week experiment of 22 lessons was prepared and two teaching methods (tradition vs geogebra aided instructions) were tested for two groups (Experimental vs Control). The data were collected through researcher made test. To find the significance means differences of the two groups, t- test was used. The study showed that there were statistically significant differences between the means of two groups of the five variables (generalization, analytical thinking, logical thinking, abstract thinking and representation).The only aspect which was found to be insignificant was the problem solving, for which the mean score of the experimental group was improved but with no statistical significance.

Keywords: Mathematical thinking, Geogebra, TPCK.

Introduction

It has been the demand of education all over the world to instill thinking behavior in students. But the important questions are what thinking is? And how we think to think? Cognitive scientists are common in this fact that thinking is the mental attribute of mind that is related to its function rather than its structure. And on the basis of this cognitive ability humans rule the whole universe and due to the same attribute, we rank the groups in different context with different labels. Additionally, different cognitive scientists defined thinking and mathematical thinking in different ways with

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respect to the function of mind. Like, in the book of Feldman (2007), thinking is the manipulation of mental representation of information. In the same way, in the book of Ruggiero (1998),"Teaching thinking across the curriculum" thinking is a mental activity or attitude of mind or habit of mind that helps in various dimensions: in the solution of a problem, searching for the truth, desire to understand, and reaching for meaning and decision making.

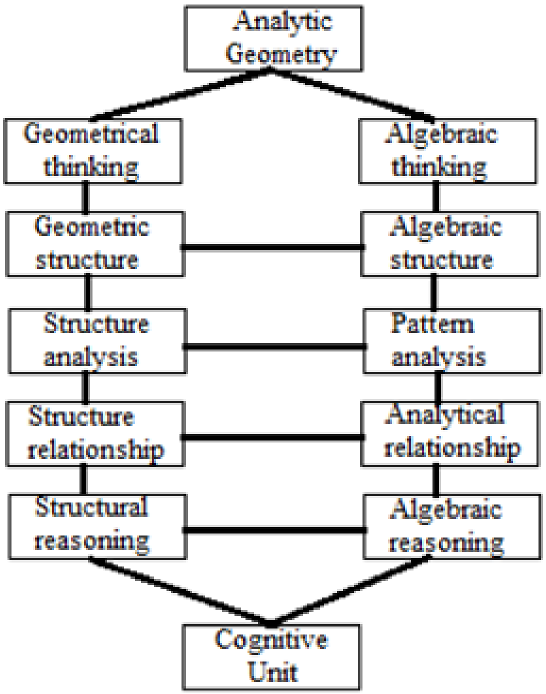
Another, thinking is the mental innate ability of human being (Vogel, 2014) and mathematical thinking develops this ability in valuable and turn it into powerful way of thinking. One must be clear in this point that thinking and mathematical thinking both are the habits of mind and both uses the same mental resources in their functional behavior. Explicitly mathematical thinking is a number of interrelated attributes which are necessary for mathematics while abstract attribute is one of the most difficult attribute in mathematical thinking (Devlin, 2000). Likewise, according to Waismann (2003), mathematical thinking is the hypothetical behavior of functional mind, and in mathematics learning we use it as a guessing and validating behaviors. So, due to this important hypothetical behavior an individual represents a concept in different semiotic systems of representation. Similarly, the importance of mathematical thinking by a distinguished mathematics educationist Stacey (2007) declared and described these mental behaviors for three ways: (1) for one of the main goal of school everywhere, (2) for learning mathematics and (3) for teaching mathematics. In conclusion, mathematical thinking is the hypothetical possibility, to the solution of a problem in hypothetical world. And the main function of mathematical thinking is to evoke our unconscious thinking to conscious, or in other simple words, mathematical thinking is like a regulator to regulate our thinking in an effective way.

There are many aspects of mathematical thinking which are strongly interconnected, it is very tough cognitive process to describe and categorize them in various ways (Karadag, 2009). Due to this reality, for assessing and the teaching of analytic geometry, a mathematical thinking model of six important aspects are categorized and described in this study. In this model, the object of analytic geometry is considered as a structure which results in different forms: expressions, equations, relations and functions. Each of these forms constitutes of variables, parameters and constants. While, in order to solve the course content of the subject analytic geometry, one must be aware of this structure which results in different forms.

Moreover, to articulate the structure of analytic geometry the following six aspects of mathematical thinking are very crucial :(1) Generalization (2) Analysis and analytical thinking (3) Logical thinking (4) Abstract thinking (5) Problem solving (6) Representation.As, algebraic concepts are everywhere in school geometry and, it is a best approach to the solution of problem in geometry by involving and understanding the concepts of variables (Dindyal, 2007). So, to learn and understand Analytic

Geometry both Algebraic and Geometric analysis of each nested results of structure is important and, the six aspects of algebraic geometric thinking model (Generalization, Analysis, Logical thinking, Abstract thinking, Problem solving, Representation) is an approach towards the understanding of analytic geometry. And to conceptualize the concept in analytic geometry, the six mathematical thinking skills of dual nature are essential and in this study, it has given the name of algebraic geometric thinking model. The model is a hybrid of Algebra and Geometry. To understand Analytic geometry both the aspects (algebraic and geometric thinking) are necessary.

Figure 1: Dual thinking behaviour in analytic geometry



For every algebraic structure, there is an equal geometrical representation in coordinate geometry. Markus Hohenwarter used the approach of Descartes and combined algebra and geometry into software language (GeoGebra). In Mathematics education, GeoGebra is being used mostly at every level.

Geogebra

Geogebra is a free of cost dynamic mathematics teaching learning software which is being used in most of the countries from primary to tertiary level. It joins three disciplines: geometry,algebra and calculus. One can easily download it through internet www.geogebra.org. (Hohen et al., 2008). Two ypes of representations (algebraic and geometric) are necessary for understanding analytic geometry (Huang, 2012, 2013). And both these issues are integrated into geogebra in a very sopfisticated way. They are labeled into this softwae as: algebra and geometry window. Through a click, the object of analytic geometry can be easily moved dynamically that cause effective understanding (Misfeldt, 2008).

Model To Integrate Geogebra Technology In Teaching of Analaytic Geometry

Description of the Model

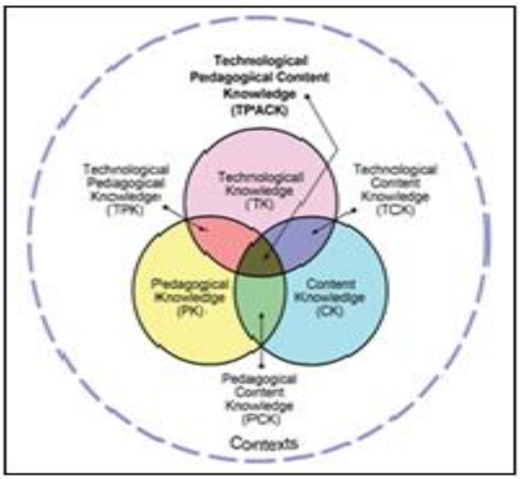
On the whole, in education, teaching of mathematics is tough and complex cognitive entity. But for effective teaching, three basic aspects: the content knowledge, the pedagogical knowledge and the pedagogical content knowledge are a license. In addition, along with these three, the teacher must know about the mathematical thinking of the students as well (Baş, et al, 2013). As, mathematics teaching is the transformation of mathematics content with concept, so to make it understandable and meaningful for the students, the teacher must know about the different aspects of its transformation. And in the process of this transformation, the teacher must know and understand the various aspects of learner: his/her misconceptions, psychological age and developmental stage as well.

Furthermore, to teach effectively, teacher must have necessary skills and competencies. In these skills, teacher should not only rely on a single content or subject matter knowledge, but he must equip himself with pedagogical content knowledge that influences teaching and content knowledge of teacher as well. In addition, Nakıboglu and Karakoç (2005) described that teacher should have the awareness of four interrelated categories: content knowledge, pedagogical knowledge, context of the learning and pedagogical content knowledge. And pedagogical content knowledge which is the intersection result of content plus pedagogy is the continuous understanding of professional development of teacher in a specific discipline.

However, the integration of technology into mathematics that change the role of teacher in the classroom does not depend on a single factor, but it takes into account different interrelated components. Gómez-Chacón and Joglar (2010), specified them in to four different components that are, cognitive, didactical, technological and affective components. To integrate technology, teacher must have a proper skill and competency. Like other profession, teaching can be learned and teacher should be a part of scholar community (Shulman, 1987).

According to UNESCO (2008), technology proficiency is necessary for the professional development and the effective integration of technology depends on the ability of teacher in technology proficiency. Ramatlapana (2014) described that the engagement of students due to technology results in higher order thinking. To implement and integrate technology in mathematics education is not so simple, teacher should have the knowledge of technological pedagogical content knowledge (TPCK) which is the extension of pedagogical content knowledge (PCK) and the teacher technological knowledge should be built around these three domains: content knowledge, pedagogical knowledge and technological knowledge. Now, how technology is to be used into the classroom to make teaching more effective. Mishra and Koehler (2006) described the three important, complex interactive domains: content, pedagogy and technology. The integration model of content, pedagogy and technology in TPCK is shown in figure 2.

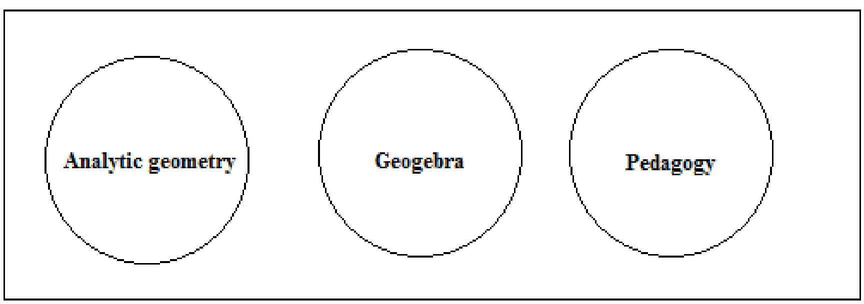
Figure 2: Intersection behavior of content, pedagogy and technology



Source: Mishra & Koehler, 2009, p. 63

The three overlapping circles results into four interrelated kinds of knowledge. In which three (PK, CK & TK) are the intersection of any two domains of content, pedagogy and technology, while the intersection of all three domains results into a new discipline of technological pedagogical content knowledge (TPACK). TPACK is the complex combination of technology with pedagogical content knowledge.To implemet geogebra into teaching of analytic geometry,during teacher training,two disciplines should be combined: Geogebra and pck for analytic geometry.

Figure 3 Three domains of Model



Analytic Geometry Content Knowledge

The organization of knowledge in the mind of teacher, understanding of structure of subject and its different representation, transformation; beyond the facts and concept of a domain (Shulman, 1986).

Geogebra Technology Knowledge

How to operate geogebra software regarding its function and tools and application. The expertise in the features of geogebra and its potential (Niess, 2006).

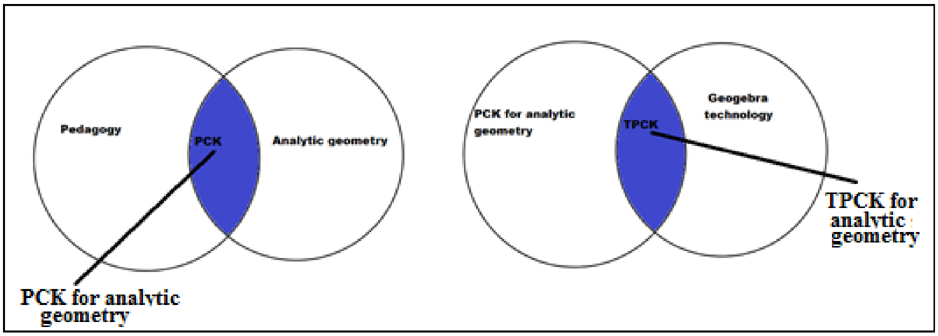
Pedagogical Knowledge

One of the effective focus elements in teacher education program that includes: the knowledge of teaching and learning process. The reflections of this are optimum engagement of students in the class. On the basis of this demarcation is drawn between professional and non-professional (Mahmood, 2014; De Miranda, 2008).

Pedagogical Analytic Geometry Knowledge

The pedagogy of how to teach and design the lesson plans for the content of analytic geometry to make teaching effective. A teacher ofanalytic geometry must know, how to make the content easier to the students. While teaching, he must use illustrations and to model the situation to daily life (Işık, Öcal, & Kar, 2013).

Figure 4: PCK & TPCK FOR analytic geometry



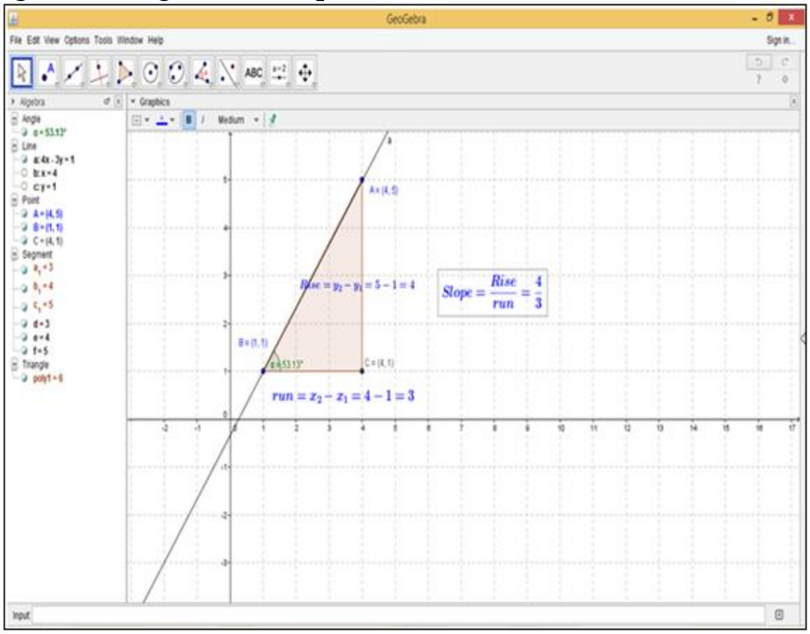
TPCK for Analytic Geometry

To teach analytic geometry through geogebra, three basic principles are require: 1) teacher must know about the content knowledge 2) pedagogical knowledge for teaching analytic geometry simply how to teach and 3) about the geogebra tools, its function, strength and applications. To combine pedagogical content knowledge of analytic geometry with geogebra technology is a complex cognitive pedagogy (Guerrero, 2010).

Through geogebra integrated lessson,the concept of slope involving different steps can be validated practically by constructing the triangle along the line. The sense of every concept can be prevailed by proper lesson planning and through correct implementation of tools of geogebra (Hohenwarter & Hohenwarter, 2008). Some basic points in planning the activity through geogebra:

a. Keeping the main objective or result of the lesson.
b. Mention the geogebra tools which will be used in the activity.
c. Design the possible action which can result the process into valid result.
d. The process of the concept should be performed through step by step.
e. Sub concept should be validated in each step and sense of understanding must be maintained in action.
f. Connect the sub concepts in scientific way.
g. Concretize and then generalize the concept.
h. The slope concept activity through geogebra is run in the below geogebra window

Figure 6: Geogebra lesson plan model structure



Objectives of the Study

The main objective of the study was to explore the significant effect of geogebra on grade-12 students in six aspects of mathematical thinking.

Sıgnıfıcance of the Study

Thinking and mathematical thinking is the pre-requisite for high grade mathematics. Both teacher and students need it in their capacity of teaching and learning. This research is useful in understanding different and dual aspects of mathematical thinking in analytic geometry in addition, the role of DGS in getting these aspects in scientific ways.

Hypothesis of the study

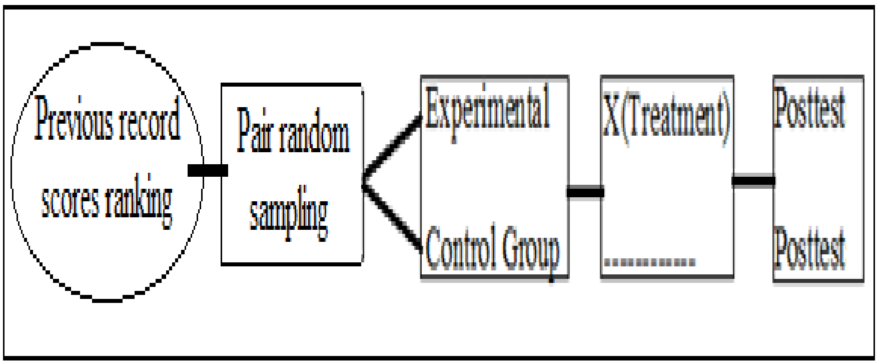
GeoGebra aided instructions do not effect significantly the mean scores of the students in each aspect of mathematical thinking post-test.

Methodology

Design of the Study

A true posttest only equivalent groups design was used because of low sample size of the total of forty students. This design is very powerful, suitable and authentic to unearth the causal effect of the small sample over random sampling. Because the random sampling in small sample there might be a chance that result into two non-equivalent random groups. In such cases, the effect might be owing to non-equivalent group rather than experimental variable. To reduce the risk, rank order should be used. And during the pair-random sampling process the students were divided and exposed to two groups, on the basis of certain characteristics which should be highly correlated with the post-test (Newby, 2014; Nestor & Schutt,2014; Ary, Jacobs, Sorensen & Walker, 2013; Cohen, Manion & Morrison, 2011).

Figure 7: Research Design



Instrument

A six week experiment of 22 lessons were prepared for this study.All the lessons were prepared by the researcher with the help of internet, GeoGebra website, GeoGebra research material (GeoGebra self-learning supporting material) and were uploaded on (http://tube.geogebra.org/mkhalilkhan). The six aspects of mathematical thinking were selected which were constructed in collaboration with mathematics educationist in the area of mathematical thinking especially with Amir Zaman and Ma 'moon (PhD doctors, 2014). And to test the hypothesis, a test of 36 questions for this study of reliability (Cronbach's Alpha coefficient 0.92) was constructed by the researcher to measure students' mathematical thinking ability(Boslaugh & Watters, 2008). İn scoring the test, rubric identify what is being assessed and it is important for the different level of an individual in response to the test item (Angelelli, 2009; Danielson & Marquez, 1998). For this research, the four point rubric scale was developed for each item. The maximum and minimum score for each item was assigned four/4 and zero/0 respectively.

Table 1
Rubric for İtem Response for Mathematical Thinking Test

Response score	*Detail description*
4	Correct answer with strong justification and correct reasoning. Using of proper reasoning and diagram and deep understanding. Process understanding.
3	Incorrect answer with consistent reasoning along with some process
2	Correct answer with inconsistent reasoning and error in mathematical process.
1	Correct option with no reasoning or justification or with a little sense of understanding in writing justification.
0	No response, completely incorrect with no reasoning

Treatment

Two teaching methods (tradition verses geogebra aided instructions) were tested on each group of 20 students. Six week experiments of 22 lessons were prepared for this study. All the lesson plans were prepared by the researcher with the help of internet, GeoGebra website, GeoGebra research material (GeoGebra self-learning supporting material) and were uploaded on (http://tube.geogebra.org/mkhalilkhan).

The experiment was started on 1st August 2014 and ended on 19th September 2014, in which two different instruction patterns were used for two groups. Both groups were taught by the same teacher under the supervision of researcher. The teacher who was volunteered for this experiment had master degrees in mathematics and education along with expertise in computer field. He was assisted 10 days training in GeoGebra learning and in implementation. The assistance was delimited to the contents: GeoGebra installation, GeoGebra menu and toolbar, navigation bar and construction protocol, slider creating, creating dynamic object, GeoGebra object properties, inserting text into graphic window, visualization of linear equation with concept, open and save GeoGebra files.

Moreover, the only difference in the two treatment patterns was that the GeoGebra aided instructions were conducted in a well-equipped computer lab. Ten computers were arranged there in U-shape and the students worked there in the pairs. On first two days, the experimental group students were trained about GeoGebra using in the two main topics: GeoGebra installation and GeoGebra user interface. While all the lessons were taught through GeoGebra applets and directed activities. The experimental group students were mostly involved in GeoGebra learning environment through drill and practice. In addition, they were engaged to learn the analytic geometry concept through applets and further they were also assigned to solve the problems with the help of GeoGebra.

Results

The data were collected by a criterion test and analyzed using SPSS.

Table 2
Descriptive statistics of experimental and control groups performance on six aspects of mathematical thinking post-test

Six Aspects of M. T	*N*	*Group*	*Mean*	*Std. Deviation*
Generalization thinking	20	Exp.	20.8	2.71
	20	Cont.	17.6	4.4
Analytical thinking	20	Exp.	20.1	2.6
	20	Cont.	15.3	4.3
Logical thinking	20	Exp.	18.6	3.3
	20	Cont.	14.7	4.6
Abstract thinking	20	Exp.	18.95	2.5
	20	Cont.	15.25	3.7
Problem solving	20	Exp.	17.2	3.6
	20	Cont.	14.9	4.2
Representation thinking	20	Exp.	20.15	2.96
	20	Cont.	14.95	4.07

In table 1, the descriptive statistics for both the groups (experimental group and control group) shows that in all six cases, the average marks obtained by the experimental group was higher than that of the control group. And at the same time if we look at the column of the standard deviation, we can see that the standard deviations of the experimental group are lower than that of the control group.

Table 3
Significance of difference for experimental and control groups on six aspects of mathematical thinking

Six Aspects of M. T	*Exp.& Cont. Groups Mean Diff*	*t-test for Equality of Means*	
		Calculated t-values	*Sig. (2-tailed)*
Generalization thinking	3.2	2.77	0.009
Analytical thinking	4.8	4.24	0.000
Logical thinking	3.9	3.09	0.004
Abstract thinking	3.7	3.69	0.001
Problem solving	2.3	1.84	0.073
Representation thinking	5.2	4.62	0.000

All t-tests in table 2 explore whether the means for each groups are same at significance level of alpha($\alpha = 0.05$). From the column of significance (i.e p-value), it can be seen that all the mathematical thinking tests were significant except problem solving as the significance value isless than 0.05. Hence it was concluded that the null hypothesis that there is no significant difference between the two groups (i.e control group and experimental group), is rejected. Therefore, it can be stated that there were significant differences between the two groups on the means of the five variables (generalization, analytical thinking, logical thinking, abstract thinking and representation). The only variable among the six variables which do not have any significant difference is the problem solving.

Discussion

Mathematical thinking is the essence for mathematical learning. Tasks, tools and environment are necessary for developing both these in every education system. Specifically, to learn analytic geometry, the integration of geogebra is the best tool in learning different aspects of this subjects as well different thinking aspects of the discipline. And, because of its open source and free in access, it can easily be used and implemented in our education system of Pakistan. But until now, the potential of this software in ability and in effectiveness has not been exposed here in Pakistan. İn this study, this software was used for the development of six aspects of mathematical thinking in analytic geometry. The result showed that, geogebra improved the six aspects (Generalization, analytical thinking, Logical thinking, Abstract thinking, Problem solving, Representation) of experimental group in comparison to control group at significant level.The findings of this research also supports the claims of different studies conducted by); Mwei et al (2011), Tran et al (2014), Bakar et al (2010), Demirbilek & Özkale (2014), Erbas & Yenmez (2011), Olkun et al (2005) and Shadaan, P. & Leong. K. E. (2013). All of them reported that the technology-integrated environments cause increase in competencies, achievement, positive attitude, mathematical reasoning and mathematical thoughts across different grades.

Conclusion

Technology should be used as a tool to support mathematics instructions along with students' mathematical thinking. Educational choices should be made on the basis of objectives, methodologies, role of the teacher and the level of the students while implementing appropriate technology. It is to add that, the geogebra applet should be built in a way that students are able to explore it independently and get a sense of concept with minimum assistance of the teacher. Geogebra is a free package and it is specially designed for the high school mathematics. It is easy to use and learn and implement in the classroom. A proper computer lab is the basic requirement to implement it in mathematics learning. So, the school and college administration need to facilitate students and teachers with respect to computer lab. Another fact is that, in this research study, the fact was to investigate the dynamic geometry software significant effect on the six aspects of M.T. In result, all the aspects of mathematical thinking except problem solving improved with significance. Because the problem solving is a tough cognitive activity, so, teacher trainers must keep them in their minds, while teachers training programs with the focus and emphasis on PCK (pedagogical content knowledge) and TPCK (technological pedagogical content knowledge).

Recommandations

Mathematics teaching should be a professional profession, and a four years program (B.S elementary and secondary mathematics education) should be implemented for pre service mathematics teachers like in turkey (METU/ Ankara). The main aim of such type of program is to develop and groom pre service teacher with a well sound understanding of how to teach mathematics and how students learn mathematics through the different ages and stages,along with confidence in the use of technology with problem solving behavior. The program should be emphasis on mathematical thinking and professional development of pre service mathematics teacher.

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APPENDIX

Scores of pretest and posttest

	Exp. Group	Cont. Group
Students	Grade-11 Pretest scores	Grade-11 Pretest scores
S_1	84	84
S_2	83	80
S_3	77	77
S_4	76	76
S_5	74	74
S_6	74	73
S_7	72	72
S_8	70	70
S_9	69	69
S_{10}	68	67
S_{11}	66	66
S_{12}	64	65
S_{13}	58	58
S_{14}	56	56
S_{15}	55	56
S_{16}	53	54
S_{17}	53	53
S_{18}	52	53
S_{19}	49	50
S_{20}	33	43